Logical statements

Notation:

- $\forall$ means "for all" or "for every"

  (universal quantifier)

- $\exists$ means "there exists"

  (existential quantifier)

- $X \Rightarrow Y$ means "if X then Y"

  "implies that"

  (conditional statement)

- $X \Leftrightarrow Y$ means "X if and only if Y"

  iff

  (i.e. $X \Rightarrow Y$ and $Y \Rightarrow X$)

  (biconditional statement)

- $\sim X$ means "not X"

  (negation)

Exs:

1) Definitions of "$A \subseteq B$" and "$A \not\subseteq B$":

$$A \subseteq B \Leftrightarrow \forall a \in A, \text{ we have } a \in B.$$

$$A \not\subseteq B \Leftrightarrow \exists a \in A \text{ such that } a \notin B.$$

"s.t."

2a) Consider the following statement:

$\forall n \in \mathbb{Z}, \exists x \in \mathbb{Q}$ s.t. $x - n = 1$.

It means "for every integer n, there exists a rational number x such that $x - n = 1$."

This is a true statement, since, for $n \in \mathbb{Z}$, we may take $x = n+1$. Then $x \in \mathbb{Q}$ and $x - n = 1$.

2b) Reversing the order of the quantified statements in the previous example gives:

$\exists x \in \mathbb{Q}$ s.t. $\forall n \in \mathbb{Z}$, $x - n = 1$.

This means "there exists a rational number x such that, for every integer n, we have $x - n = 1$."

This is a false statement. To see why, note that for $x \in \mathbb{Q}$, if we take n to be the smallest integer which is greater than or equal x, then $x - n \le 0$, so $x - n \neq 1$.

Helpful facts:

1) The statement $X \Rightarrow Y$ is equivalent to the statement $\sim Y \Rightarrow \sim X$. (contrapositive)

Exs:

- "If $n \in \mathbb{N}$ then $n \in \mathbb{Q}$."

  Conditional statement:
  $$n \in \mathbb{N} \Rightarrow n \in \mathbb{Q}$$
  Contrapositive:
  $$n \notin \mathbb{Q} \Rightarrow n \notin \mathbb{N}$$
  (both true)

- "If $x \in \mathbb{R}$ then $x \in \mathbb{Q}$."

  Conditional statement:
  $$x \in \mathbb{R} \Rightarrow x \in \mathbb{Q}$$
  Contrapositive:
  $$x \notin \mathbb{Q} \Rightarrow x \notin \mathbb{R}$$
  (both false)

2a) Negation of a universally quantified statement:

$\sim(\forall x \in A, X)$ (the negation of "for all $x \in A$, statement X holds")

is equivalent to

$\exists x \in A$ s.t. $\sim X$ (there exists an $x \in A$ such that statement X does not hold)

b) Negation of an existentially quantified statement:

$\sim(\exists x \in A$ s.t. $X)$ (the negation of "there exists an $x \in A$ s.t. statement X holds")

is equivalent to

$\forall x \in A, \sim X$ (for every $x \in A$, statement X does not hold)

Exs:

1) Example from before:

$\exists x \in \mathbb{Q}$ s.t. $\underbrace{(\forall n \in \mathbb{Z},\ \ x-n=1)}_{X}$. (false)

Negation:

$\forall x \in \mathbb{Q},\ \sim X$

Equivalently:

$\forall x \in \mathbb{Q},\ \exists n \in \mathbb{Z}$ s.t. $x-n \neq 1$. (true)

2) Prove or disprove:

$\forall t \in \mathbb{R}, \underbrace{(\exists n \in \mathbb{Z} \text{ s.t. } nt>t)}_{X}$.

Scratch work / thought process:

First try some exs. and decide whether you think the statement is going to be true <u>for every</u> $t \in \mathbb{R}$:

- If $t>0$ we can take $n=2$, then $nt>t$. ✓
- If $t<0$ we can take $n=-1$ then $nt>t$. ✓
- But... what about if $t=0$? ...

Negation:

$\exists t \in \mathbb{R}$ s.t. $\sim X$

Equivalently:

$\exists t \in \mathbb{R}$ s.t. $\forall n \in \mathbb{Z}$, $\sim (nt > t)$.

$\exists t \in \mathbb{R}$ s.t. $\forall n \in \mathbb{Z}$, $nt \leq t$. (true)

Pf. that this is true: Let $t = 0$. Then $\forall n \in \mathbb{Z}$,

$nt = 0 = t$, so $nt \leq t$. $\blacksquare$

Since the negation of the original statement is true, the original statement itself is false.